

**MUMBAI UNIVERSITY CBCGS**

**APPLIED MATHEMATICS I MAY 2019 PAPER SOLUTIONS**

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**Q1)a) If  $u = \log\left(\frac{x}{y}\right) + \log\left(\frac{y}{x}\right)$ , find  $\frac{\partial u}{\partial x}$  and  $\frac{\partial u}{\partial y}$ . (3M)**

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0 \cdot u = 0$$

**Ans :**  $\frac{\partial u}{\partial x} = \frac{-y}{x}$   
 $\frac{\partial u}{\partial y} = \frac{y}{x}$

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**Q1)b) Find the value of  $\tanh(\log x)$  if  $x = \sqrt{3}$ . (3M)**

**Ans :** Let

$$\begin{aligned} z &= \tanh(\log \sqrt{3}) \\ \therefore \tanh^{-1} z &= \log \sqrt{3} \\ \therefore \frac{1}{2} \log\left(\frac{1+z}{1-z}\right) &= \frac{1}{2} \log 3 \\ \therefore \log\left(\frac{1+z}{1-z}\right) &= \log \sqrt{3} \end{aligned}$$

By componendo and dividendo

$$\frac{2}{-2z} = \frac{3+1}{1-3}$$

$$\therefore z = \frac{3-1}{3+1}$$

$$\therefore \tanh \log \sqrt{3} = \frac{3-1}{3+1} = \frac{1}{2}$$

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**Q1)c) Evaluate  $\lim_{x \rightarrow 3} \left[ \frac{1}{x-3} - \frac{1}{\log(x-2)} \right]$ . (3M)**

**Ans:**  $\lim_{x \rightarrow 3} \left[ \frac{1}{x-3} - \frac{1}{\log(x-2)} \right] [\infty - \infty] = \lim_{x \rightarrow 3} \frac{\log(x-2) - (x-3)}{(x-3)\log(x-2)} \frac{0}{0}$

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$$= \lim_{x \rightarrow 3} \frac{\frac{1}{x-2} - 1}{\log(x-2) + \frac{(x-3)}{(x-2)}} = \lim_{x \rightarrow 3} \frac{-x+3}{(x-2)\log(x-2) + (x-3)} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \lim_{x \rightarrow 3} \frac{-1}{\frac{(x-2)}{(x-2)} + \log(x-2) + 1} = -\frac{1}{2}.$$

**Q1)d) If  $u = r^2 \cos 2\theta, v = r^2 \sin 2\theta$ , find  $\frac{\partial(u,v)}{\partial(r,\theta)}$ . (3M)**

**Ans:** We have  $\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(r,\theta)}$ . But  $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & a \\ b & -b \end{vmatrix} = -2ab$

And  $\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} 2r \cos 2\theta & -2r^2 \sin 2\theta \\ 2r \sin 2\theta & 2r^2 \cos 2\theta \end{vmatrix} = 4r^3.$

**Q1)e) Express the matrix  $A = \begin{pmatrix} 2+3i & 2 & 3i \\ -2i & 0 & 1+2i \\ 4 & 2+5i & -i \end{pmatrix}$  as the sum of a Hermitian and a Skew-Hermitian matrix. (4M)**

**Hermitian matrix.**

**Ans:** We have

$$A' = \begin{bmatrix} 2+3i & -2i & 4 \\ 2 & 0 & 2+5i \\ 3i & 1+2i & -i \end{bmatrix}$$

$$\therefore A^\theta = (\bar{A}') = \begin{bmatrix} 2-3i & 2i & 4 \\ 2 & 0 & 2-5i \\ -3i & 1-2i & i \end{bmatrix}$$

$$\therefore A + A^\theta = \begin{bmatrix} 4 & 2+2i & 4+3i \\ 2-2i & 0 & 3-3i \\ 4-3i & 3+3i & 0 \end{bmatrix}$$

$$A - A^\theta = \begin{bmatrix} 6i & 2-2i & -4+3i \\ -2-2i & 0 & -1+7i \\ 4+3i & 1+7i & -2i \end{bmatrix}$$

Let  $P = \frac{1}{2}(A + A^\theta), Q = \frac{1}{2}(A - A^\theta).$

But, we know that P is Hermitian and Q is Skew-Hermitian and  $A = P + Q$ .

$$\therefore A = P + Q = \begin{bmatrix} 2 & 1+i & (4+3i)/2 \\ 1-i & 0 & (3-3i)/2 \\ (4-3i)/2 & (3+3i)/2 & 0 \end{bmatrix} + \begin{bmatrix} 3i & 1-i & (-4+3i)/2 \\ -1-i & 0 & (-1+7i)/2 \\ (4+3i)/2 & (1+7i)/2 & -i \end{bmatrix}.$$

**Q1)f) Expand  $\tan^{-1} x$  in powers of  $\left(x - \frac{\pi}{4}\right)$ . (4M)**

**Ans:** Let

$$f(x) = \tan^{-1} x, a = \frac{\pi}{4}$$

$$\therefore f(x) = \tan^{-1} x, f'(x) = \frac{1}{1+x^2}, f''(x) = \frac{-2x}{(1+x^2)^2}$$

$$\therefore f\left(\frac{\pi}{4}\right) = \tan^{-1}\left(\frac{\pi}{4}\right), f'\left(\frac{\pi}{4}\right) = \frac{1}{1+\left(\frac{\pi}{4}\right)^2}, f''\left(\frac{\pi}{4}\right) = -\frac{\frac{\pi}{2}}{\left[1+\left(\frac{\pi^2}{16}\right)\right]^2}, \text{etc}$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$$

$$\therefore \tan^{-1} x = \tan^{-1}\left(\frac{\pi}{4}\right) + \left(x - \frac{\pi}{4}\right) \cdot \frac{1}{1+\left(\frac{\pi}{4}\right)^2} - \left(\frac{\pi}{4}\right) \left(x - \frac{\pi}{4}\right)^2 \cdot \frac{1}{\left[1+\left(\frac{\pi^2}{16}\right)\right]^2} + \dots$$

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**Q2)a) Expand  $\sin^7 \theta$  in a series of sines of multiples of  $\theta$  .**

**(6M)**

$$x = \cos \theta + i \sin \theta$$

$$\therefore \frac{1}{x} = \cos \theta - i \sin \theta$$

$$x + \frac{1}{x} = 2 \cos \theta$$

$$x - \frac{1}{x} = 2i \sin \theta$$

$$x^n = \cos n\theta + i \sin n\theta$$

$$\frac{1}{x^n} = \cos n\theta - i \sin n\theta$$

$$\therefore x^n + \frac{1}{x^n} = 2 \cos n\theta$$

$$x^n - \frac{1}{x^n} = 2i \sin n\theta$$

Now, by Binomial Theorem

$$(2i \sin \theta)^7 = \left(x - \frac{1}{x}\right)^7 = x^7 - 7x^6 \cdot \frac{1}{x} + 21x^5 \cdot \frac{1}{x^2} - 35x^4 \cdot \frac{1}{x^4} + 21x^2 \cdot \frac{1}{x^5} - 7x \cdot \frac{1}{x^6} + \frac{1}{x^7}$$

$$\therefore (2i \sin \theta)^7 = x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}$$

$$\therefore (2i \sin \theta)^7 = \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right)$$

$$\therefore (2i \sin \theta)^7 = 2i \sin 7\theta - 7 \cdot (2i \sin 5\theta) + 21 \cdot (2i \sin 3\theta) - 35 \cdot (2i \sin \theta)$$

$$\therefore -2^6 \sin^7 \theta = \sin 7\theta - 7 \sin 5\theta + 21 \sin 3\theta - 35 \sin \theta$$

$$\therefore \sin^7 \theta = \frac{-1}{2^6} (\sin 7\theta - 7 \sin 5\theta + 21 \sin 3\theta - 35 \sin \theta)$$

**Q2)b) If  $y = \sin^2 x \cos^3 x$  , then find  $y_n$  .**

**(6M)**

**Ans:** We have

$$y = \sin^2 x \cos^3 x$$

$$\therefore y = \sin^2 x \cos^2 x \cdot \cos x = \frac{1}{4} (\sin 2x)^2 \cos x$$

$$\therefore y = \frac{1}{8} (1 - \cos 4x) \cos x = \frac{1}{8} (\cos x - \cos 4x \cos x)$$

$$\therefore y = \frac{1}{8} \cos x - \frac{1}{16} (\cos 5x + \cos 3x)$$

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By using the result  $y_n = a^n \cos\left(ax + \frac{n\pi}{2}\right)$

$$y_n = \frac{1}{8} \cos\left(x + \frac{n\pi}{2}\right) - \frac{1}{16} \cdot 5^n \cos\left(5x + \frac{n\pi}{2}\right) - \frac{1}{16} \cdot 3^n \cos\left(3x + \frac{n\pi}{2}\right)$$

**Q2)c) Find the stationary values of  $x^3 + y^3 - 3axy, a > 0$  .**

**(8M)**

**Ans:** We have  $f(x, y) = x^3 + y^3 - 3axy$ .

Step 1:

$$f_x = 3x^2 - 3ay, f_y = 3y^2 - 3ax$$
$$r = f_{xx} = 6x, s = f_{xy} = -3a, t = f_{yy} = 6y$$

Step 2: We now solve,

$$f_x = 0, f_y = 0$$
$$\therefore x^2 - ay = 0, y^2 - ax = 0$$

To eliminate y, we put  $y = \frac{x^2}{a}$  in the second equation .

$$\therefore x^4 - a^3x = 0,$$
$$\therefore x(x^3 - a^3) = 0$$

Hence,  $x=0$  or  $x=a$

When  $x=0, y=0$  and when  $x=a, y=a$ .

Hence,  $(0,0)$  and  $(a,a)$  are stationary points.

Step 3: (i) For  $x=0, y=0, r = f_{xx} = 0, s = f_{xy} = -3a, t = f_{yy} = 0$  .

Hence,  $rt - s^2 = 0 - 9a < 0$  .

Hence,  $f(x,y)$  is neither maximum nor minimum. It is a saddle point.

(ii) For  $x=a, y=a$ ,

$$r = f_{xx} = 6a, s = f_{xy} = -3a, t = f_{yy} = 6a$$
$$\therefore rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0 \quad \text{Hence, } f(x,y) \text{ is stationary at } x=a, y=a .$$

And  $r = f_{xx} = 6a > 0, \therefore a > 0$

Hence  $f(x,y)$  is minimum at  $x=a, y=a$  .

Putting  $x=a, y=a$  in  $x^3 + y^3 - 3axy$ , the minimum value of

$$f(x, y) = a^3 + a^3 - 3a^3 = -a^3 .$$

**Q3)a) Compute the real root of  $x \log_{10}^x - 1.2 = 0$  correct to three places of decimals using Newton-Raphson method. (6M)**

**Ans:** We first note that  $f(x) = x \log_{10}^x - 1.2$  .

$$\therefore f(1) = 1 \log_{10}^1 - 1.2 = -1.2, f(2) = 2 \log_{10}^2 - 1.2 = -0.5979$$

$$f(3) = 3 \log_{10}^3 - 1.2 = 0.2313$$

Since  $f(x)$  changes its sign from negative to positive as  $x$  goes from 2 to 3, there is a root between 2 and 3.

$$\text{Now, } f'(x) = x \cdot \frac{1}{x \log_e^{10}} + \log_{10}^x = (\log_e^{10})^{-1} + \log_{10}^x = 0.4343 + \log_{10}^x$$

Hence, by Newton-Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0, 1, 2, 3, \dots$$

$$= x_n - \frac{x \log_{10}^x - 1.2}{0.4343 + \log_{10}^x}$$

$$\text{For } x_0 = 3, \quad x_1 = 3 - \frac{3 \log_{10}^3 - 1.2}{0.4343 + \log_{10}^3} = 2.74615$$

$$\text{For } x_1 = 2.74615, \quad x_2 = 2.74615 - \frac{(2.74615) \cdot \log(2.74615) - 1.2}{0.4343 + \log 2.74615} = 2.7406 .$$

$$\text{For } x_2 = 2.7406, \quad x_3 = 2.7406 \quad \text{Hence } x = 2.7406 .$$

**Q3)b) Show that the system of equations**

$2x - 2y + z = \lambda x, 2x - 3y + 2z = \lambda y, -x + 2y = \lambda z$  can possess a non-trivial solution only if

$\lambda = 1, \lambda = -3$  . **Obtain the general solution in each case. (6M)**

$$\text{Ans: We have } \begin{pmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system has non-trivial solution if the rank of A is less than the number of unknowns.

The rank of A will be less than three if  $|A|=0$ .

$$\text{Now, } \begin{vmatrix} 2-\lambda & -2 & 1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$$

$$\therefore (2-\lambda)(\lambda^2 + 3\lambda - 4) + 2(-2\lambda + 2) + 1(4 - 3 - \lambda) = 0$$

$$\therefore (2-\lambda)(\lambda+4)(\lambda-1) - 4(\lambda-1) - (\lambda-1) = 0$$

$$\therefore (\lambda-1)[2\lambda+8-\lambda^2-4\lambda-4-1] = 0$$

$$\therefore (\lambda-1)(-\lambda^2-2\lambda+3) = 0$$

$$\therefore (\lambda-1)(\lambda-1)(\lambda+3) = 0$$

$$\therefore \lambda = 1, \lambda = -3$$

(i) If  $\lambda=1$ , we have,

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

By  $R_2 \rightarrow R_2 - 2R_1$ ,  $R_3 \rightarrow R_3 + R_1$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore x - 2y + z = 0$ . Putting  $z = t_1$ ,  $y = t_2$ .

$\therefore$  The solution is  $x = 2t_2 - t_1$ ,  $y = t_2$ ,  $z = t_1$ .

**Q3)c) If  $\tan(\alpha + i\beta) = \cos \theta + i \sin \theta$ , prove that  $\alpha = \frac{n\pi}{2} + \frac{\pi}{4}$  and  $\beta = \frac{1}{2} \log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ .**

**(8M)**

**Ans:** We have

$$\tan(\alpha + i\beta) = \cos \theta + i \sin \theta$$

$$\therefore \tan(\alpha - i\beta) = \cos \theta - i \sin \theta$$

$$\therefore \tan 2\alpha = \tan[(\alpha + i\beta) + (\alpha - i\beta)]$$

$$= \frac{\tan(\alpha + i\beta) + \tan(\alpha - i\beta)}{1 - \tan(\alpha + i\beta) \cdot \tan(\alpha - i\beta)} = \frac{2 \cos \theta}{1 - (\cos^2 \theta + \sin^2 \theta)}$$

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$$\therefore \tan 2\alpha = \frac{2 \cos \theta}{0}$$

$$\therefore 2\alpha = \frac{\pi}{2}$$

$$2\alpha = n\pi + \frac{\pi}{2}$$

$$\therefore \alpha = \frac{n\pi}{2} + \frac{\pi}{4}$$

$$\begin{aligned} \tan(2i\beta) &= \tan[(\alpha + i\beta) - (\alpha - i\beta)] \\ &= \frac{\tan(\alpha + i\beta) - \tan(\alpha - i\beta)}{1 + \tan(\alpha + i\beta) \cdot \tan(\alpha - i\beta)} = \frac{2i \sin \theta}{1 + 1} = i \sin \theta \end{aligned}$$

$$\therefore i \tan(h 2\beta) = i \sin \theta$$

$$\therefore \tan(h 2\beta) = \sin \theta$$

$$\therefore 2\beta = \tanh^{-1}(\sin \theta) = \frac{1}{2} \log \left( \frac{1 + \sin \theta}{1 - \sin \theta} \right)$$

But

$$1 + \sin \theta = \left( \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} \right) + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \left( \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right)^2$$

$$1 - \sin \theta = \left( \sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} \right) - 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \left( \cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right)^2$$

$$\therefore 2\beta = \frac{1}{2} \log \left[ \frac{\left[ \cos \left( \frac{\theta}{2} \right) + \sin \left( \frac{\theta}{2} \right) \right]^2}{\left[ \cos \left( \frac{\theta}{2} \right) - \sin \left( \frac{\theta}{2} \right) \right]^2} \right] = \log \left[ \frac{\left[ \cos \left( \frac{\theta}{2} \right) + \sin \left( \frac{\theta}{2} \right) \right]}{\left[ \cos \left( \frac{\theta}{2} \right) - \sin \left( \frac{\theta}{2} \right) \right]} \right]$$

$$\therefore \beta = \frac{1}{2} \log \left[ \frac{1 + \tan \left( \frac{\theta}{2} \right)}{1 - \tan \left( \frac{\theta}{2} \right)} \right] = \frac{1}{2} \log \tan \left( \frac{\pi}{4} + \frac{\theta}{2} \right)$$

**Q4)a) Using the encoding matrix as  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , encode and decode the message MOVE. (6M)**

**Ans:** Step 1: To replace letters by numbers

M	O	V	E
13	15	22	5

We write this in a sequence of 2 X 2 matrix



$$\begin{bmatrix} 13 \\ 15 \end{bmatrix} \begin{bmatrix} 22 \\ 5 \end{bmatrix}.$$

Step 2 : To encode the message

We now premultiply each of the above column-vectors by encoding matrix  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ .

$$\therefore \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 13 & 22 \\ 15 & 5 \end{bmatrix} = \begin{bmatrix} 13+15 & 22+5 \\ 0+15 & 0+5 \end{bmatrix} = \begin{bmatrix} 28 & 27 \\ 15 & 5 \end{bmatrix}.$$

The above message is transmitted in the following linear form taking numbers column-wise. The message is transmitted in the linear form as

28    15    27    5

Step 3: To decode the message :

The above received message is now written in a sequence of 2 X 1 column matrix as

$$\begin{bmatrix} 28 & 27 \\ 15 & 5 \end{bmatrix}.$$

The above matrix is then premultiplied by the inverse of the coding matrix i.e., by

$$\therefore \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 28 & 27 \\ 15 & 5 \end{bmatrix} = \begin{bmatrix} 28-15 & 27-5 \\ 0+15 & 0+5 \end{bmatrix} = \begin{bmatrix} 13 & 22 \\ 15 & 5 \end{bmatrix}.$$

Step 4 : To replace numbers by letters

The columns of this matrix are written in linear form as

14    15    23    27    19    20    21    4    25    27

Now it is transformed into letters using corresponding alphabets

14    15    23    27    19    20    21    4    25    27  
N    O    W    \*    S    T    U    D    Y    \*

This is the required message.

**Q4)b) If  $u = f(e^{x-y}, e^{y-z}, e^{z-x})$  then prove that  $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$  .** **(6M)**

**Ans:** Let  $X = e^{x-y}, Y = e^{y-z}, Z = e^{z-x}$  . Then  $u = f(X, Y, Z)$

$$\begin{aligned} \therefore \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial x} \\ \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial X} e^{x-y}(1) + \frac{\partial u}{\partial Y} (0) + \frac{\partial u}{\partial Z} e^{z-x}(-1) \\ \therefore \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial X} e^{x-y} - \frac{\partial u}{\partial Z} e^{z-x} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial y} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial X} e^{x-y}(-1) + \frac{\partial u}{\partial Y} e^{y-z}(1) + \frac{\partial u}{\partial Z} (0) \\ \therefore \frac{\partial u}{\partial y} &= -\frac{\partial u}{\partial X} e^{x-y} + \frac{\partial u}{\partial Y} e^{y-z} \\ \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial X} \cdot \frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y} \cdot \frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z} \cdot \frac{\partial Z}{\partial z} \\ \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial X} (0) + \frac{\partial u}{\partial Y} e^{y-z}(-1) + \frac{\partial u}{\partial Z} e^{z-x}(1) \\ \therefore \frac{\partial u}{\partial z} &= -\frac{\partial u}{\partial Y} e^{y-z} + \frac{\partial u}{\partial Z} e^{z-x} \end{aligned}$$

**Q4)c) If  $y = a \cos(\log x) + b \sin(\log x)$  , then show that  $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2 + 1)y_n = 0$**  **(8M)**

**Ans:** We have

$$\begin{aligned} y &= a \cos(\log x) + b \sin(\log x) \\ \therefore y_1 &= -a \sin(\log x) \cdot \frac{1}{x} + b \cos(\log x) \cdot \frac{1}{x} \\ \therefore xy_1 &= -a \sin(\log x) + b \cos(\log x) \end{aligned}$$

Differentiating again w.r.t  $x$  ,

$$\begin{aligned} \therefore xy_2 + y_1 &= -a \cos(\log x) \cdot \frac{1}{x} - b \sin(\log x) \cdot \frac{1}{x} \\ \therefore x^2 y_2 + xy_1 + y &= 0 \end{aligned}$$

Applying Leibnitz's theorem to each term, we get

$$x^2 y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2!}(2)y_n + [xy_{n+1} + n(1)y_n] + y_n = 0$$

$$\therefore x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2 - n + n + 1)y_n = 0$$

$$\therefore x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2 + 1)y_n = 0$$

**Q5)a) If  $1, \alpha, \alpha^2, \alpha^3, \alpha^4$  are the roots of  $x^5 - 1 = 0$ , find them and show that**

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4)=5.$$

**(6M)**

**Ans:** We have

$$x^5 = 1 = \cos 0 + i \sin 0$$

$$\therefore x^5 = \cos(2k\pi + i \sin(2k\pi))$$

$$\therefore x = (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{5}} = \cos \frac{2k\pi}{5} + i \sin \frac{2k\pi}{5}$$

Putting  $k = 0, 1, 2, 3, 4$ , we get the five roots as

$$x_0 = \cos 0 + i \sin 0$$

$$\therefore x_1 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}, x_2 = \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}, x_3 = \cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}, x_4 = \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}$$

Putting  $x_1 = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5} = \alpha$  we see that  $x_2 = \alpha^2, x_3 = \alpha^3, x_4 = \alpha^4$ .

Therefore, the roots are  $1, \alpha, \alpha^2, \alpha^3, \alpha^4$  and hence

$$x^5 - 1 = (x-1)(x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\therefore \frac{x^5 - 1}{x - 1} = (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4)$$

$$\therefore (x-\alpha)(x-\alpha^2)(x-\alpha^3)(x-\alpha^4) = x^4 + x^3 + x^2 + x + 1$$

Putting  $x = 1$ , we get

$$(1-\alpha)(1-\alpha^2)(1-\alpha^3)(1-\alpha^4) = 5$$

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Q5)b) If  $\theta = t^n e^{\frac{-r^2}{4t}}$ ,

Find n which will make  $\frac{\partial \theta}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \theta}{\partial r} \right)$ .

(6M)

Ans:

$$\begin{aligned} \frac{\partial \theta}{\partial t} &= n t^{n-1} \cdot e^{\frac{-r^2}{4t}} + t^n e^{\frac{-r^2}{4t}} \cdot \left( \frac{-r^2}{4} \right) \left( \frac{-1}{t^2} \right) \\ &= \frac{n}{t} \cdot t^n \cdot \frac{\theta}{t^n} + t^n \cdot \frac{\theta}{t^n} \left( \frac{r^2}{4t^2} \right) \\ &= \frac{n}{t} \theta + \frac{r^2}{4t^2} \theta = \left( \frac{n}{t} + \frac{r^2}{4t^2} \right) \theta \\ &\left( \because e^{\frac{-r^2}{4t}} = \frac{\theta}{t^n} \right) \end{aligned}$$

Also,

$$\begin{aligned} \frac{\partial \theta}{\partial r} &= t^n \cdot e^{\frac{-r^2}{4t}} \cdot \left( \frac{-2r}{4t} \right) = \frac{-r\theta}{2t} \\ \therefore r^2 \frac{\partial r}{\partial \theta} &= \frac{-r^3 \theta}{2t} \\ \therefore \frac{\partial}{\partial r} \left( r^2 \frac{\partial \theta}{\partial r} \right) &= \frac{\partial}{\partial r} \left( \frac{-r^3 \theta}{2t} \right) = \frac{-1}{2t} \cdot \frac{\partial}{\partial r} (r^3 \theta) = \frac{-1}{2t} \left[ r^3 \frac{\partial \theta}{\partial r} + 3r^2 \theta \right] \\ &= \frac{-1}{2t} \left[ \frac{-r^4 \theta}{2t} + 3r^2 \theta \right] = r^2 \left( \frac{r^2}{4t^2} - \frac{3}{2t} \right) \theta \\ \therefore \frac{1}{r^2} \cdot \frac{\partial}{\partial r} \left( r^2 \frac{\partial \theta}{\partial r} \right) &= \left( \frac{r^2}{4t^2} - \frac{3}{2t} \right) \theta \end{aligned}$$

Now equating we get

$$\begin{aligned} \frac{n}{t} &= \frac{-3}{2t} \\ \therefore n &= \frac{-3}{2} \end{aligned}$$

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**Q5)c) Find the root (correct to three places of decimals) of  $x^3 - 4x + 9 = 0$  lying between 2 and 3 by using Regula-Falsi method . (8M)**

**Ans:** Let  $y = f(x) = x^3 - 4x - 9$  . Here,  $x_1 = 2$  and  $x_2 = 3$ .

$$\therefore y_1 = f(x_1) = f(2) = 2^3 - 4(2) - 9 = 8 - 8 - 9 = -9 < 0$$

$$y_2 = f(x_2) = f(3) = 3^3 - 4(3) - 9 = 27 - 12 - 9 = 6 > 0$$

Since  $f(x)$  changes its sign from negative to positive as  $x$  goes from 2 to 3 , there is a root between 2 and 3 .

The root is given by

$$\therefore x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2(6) - 3(-9)}{6 - (-9)} = \frac{39}{15} = 2.6 .$$

Now,  $y_3 = f(x_3) = f(2.6) = (2.6)^3 - 4(2.6) - 9 = -1.82 > 0$

Since  $f(x)$  changes its sign from negative to positive as  $x$  goes from 2.6 to 3 , there is a root between 2.6 and 3 .

First Iteration : Let

$$x_1 = 2.6, x_2 = 3, y_1 = -1.82, y_2 = 6$$

$$\therefore x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2.6(6) - 3(-1.82)}{6 - (-1.82)} = 2.693$$

$$y_3 = f(x_3) = (2.693)^3 - 4(2.693) - 9 = -0.242 > 0$$

Since  $f(x)$  changes its sign from negative to positive as  $x$  goes from 2.693 to 3 , there is a root between 2.693 to 3 .

Second Iteration : Let

$$x_1 = 2.693, x_2 = 3, y_1 = -0.262, y_2 = 6$$

$$\therefore x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{2.693(6) - 3(-0.262)}{6 - (-0.262)} = 2.7058 = 2.706$$

$$y_3 = f(x_3) = (2.706)^3 - 4(2.706) - 9 = -0.009 > 0$$

Since  $f(x)$  changes its sign from negative to positive as  $x$  goes from 2.706 to 3 , there is a root between 2.706 to 3 .

Third Iteration : Let

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$$x_1 = 3, x_2 = 2.706, y_1 = 6, y_2 = -0.009$$

$$\therefore x_3 = \frac{x_1 y_2 - x_2 y_1}{y_2 - y_1} = \frac{3(0.03) - 2.706(6)}{-0.009 - 6} = 2.706$$

Hence, the root correct to three places of decimals = 2.706 .

**Q6)a) Find non-singular matrices P and Q such that**  $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$  **is reduced to normal**

**form. Also find its rank.**

**(6M)**

**Ans:** We first write  $A = I_3 A I_4$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

By  $R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ ? & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ ? & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ ? & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ ? & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

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$$\therefore \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = PAQ .$$

Hence , the rank of A is 2 .

**Q6)b) Find the principle value of  $(1+i)^{1-i}$  .**

**(6M)**

**Ans: Let**

$$z = (1+i)^{1-i}, \therefore \log z = (1-i)\log(1+i)$$

$$\therefore \log z = (1-i)\left[\log\sqrt{1+1} + i \tan^{-1}1\right]$$

$$= (1-i)\left[\frac{1}{2}\log 2 + i \cdot \frac{\pi}{4}\right] = \frac{1}{2}\log 2 + \frac{i\pi}{4} - \frac{i}{2}\log 2 + \frac{\pi}{4}$$

$$= \left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right) + i\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) = x + iy$$

$$\therefore z = e^{x+iy} = e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$$

$$= e^{\left(\frac{1}{2}\right)\log 2 + \left(\frac{\pi}{4}\right)} \left[ \cos\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) + i \sin\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) \right]$$

$$= \sqrt{2}e^{\frac{\pi}{4}} \left[ \cos\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) + i \sin\left(\frac{\pi}{4} - \frac{1}{2}\log 2\right) \right]$$

**Q6)c) Solve the following equations by Gauss-Seidel method.**

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

**(Take three iterations)**

**(8M)**

**Ans:** We first write the three equations as

$$x = \frac{1}{27}(85 - 6y + z)$$

$$y = \frac{1}{15}(72 - 6x - 2z)$$

$$z = \frac{1}{54}(110 - x - y)$$

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(i) First Iteration : We start with the approximation  $y=0, z=0$  and we get

$$\therefore x_1 = \frac{85}{27} = 3.15 .$$

We use this approximation to find  $y_1$  i.e. we put  $x = 3.15, z=0$  in the second equation

$$\therefore y_1 = \frac{1}{15}[72 - 6(3.15)] = 3.54 .$$

We use these values of  $x_1$  and  $y_1$  to find  $z_1$  i.e. we put  $x=3.15$  and  $y=3.54$  in the third equation

$$\therefore z_1 = \frac{1}{54}(110 - 3.15 - 3.54) = 1.91 .$$

(ii) Second Iteration : We use the latest values of  $y$  and  $z$  to find  $x$ , i.e. we put  $y= 3.54, z=1.91$  in equation 1, we get

$$x_2 = \frac{1}{27}[85 - 6(3.54) + 1.91] = 2.43$$

We put  $x = 2.43, z = 1.91$  to find  $y$  from equation 2. Thus ,

$$y_2 = \frac{1}{15}[72 - 6(2.43) - 2(1.91)] = 3.57$$

We put  $x = 2.43, y = 3.57$  in equation 3 to find  $z$ . Thus ,

$$z_2 = \frac{1}{54}[110 - 2.43 - 3.57] = 1.93$$

(iii) Third iteration: Putting  $y = 3.57, z = 1.93$  in equation (1) we get

$$x_3 = \frac{1}{27}[85 - 6(3.57) + 1.93] = 2.43$$

Putting  $x = 2.43, z = 1.93$  in equation 2 we get

$$y_3 = \frac{1}{15}[72 - 6(2.43) - 2(1.93)] = 3.57$$

Putting  $x=2.43, y=3.57$  in equation 3 we get

$$z_3 = \frac{1}{54}[110 - 2.43 - 3.57] = 1.93 .$$

Since the second and third iteration give the same values

$$x = 2.43, y = 3.57, z = 1.93$$

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